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Stability of Difference Methods for Initial-Value Type Partial Differential Equations[1]

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ABSTRACT

A systematic way to formulate stable difference expressions for initial and mixed boundary value problems is presented. It is shown that difference schemes must satisfy "invariants" in order to be stable, consistent approximations to the partial differential equation. An open loop technique is presented to choose a stable time step, Δt.

I. INTRODUCTION

The stability analysis of difference schemes for pure initial-value type problems is well developed [*1*], but this theory is restricted to constant coefficient, linear partial differential equations. There is no corresponding analysis for difference approximations to variable coefficient, mixed boundary-initial-value type. In fact, there does not exist, for either class of equations, a systematic way to choose the difference approximations. This paper shows that an expansion of the Schur–Cohn determinants determines the type of difference schemes which should be used as well as provides an open loop (as opposed to iterative) method to evaluate stability of the difference scheme during the course of the calculation. "Evaluating the stability" of the difference equations for varying coefficients is heuristic, but it is often the only alternative when calculations must be performed for varying coefficient partial differential equations.

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II. The Initial Value Problem

Consider the nth order linear, constant coefficient partial differential equation defined on R: $\{0 \leqslant t \leqslant T, -\infty < x < \infty\}$:

$$\frac{\partial^n u}{\partial t^n} - \alpha'_{n-1} \frac{\partial^{n-1} u}{\partial t^{n-1}} - \alpha'_{n-2} \frac{\partial^{n-2} u}{\partial t^{n-2}} \cdots$$

$$-\alpha'_0 u - \beta_{n-1} \frac{\partial^{n-1} u}{\partial t^{n-1}} - \beta_{n-2} \frac{\partial^{n-2} u}{\partial t^{n-2}} \cdots -\beta\, u = 0, \tag{1}$$

where the α'_i are the sum of several differential operators $(\partial/\partial x)$ with respect to the independent space variable x, which is treated here as a single independent variable. The β_i are constants. The problem is: given the n initial conditions $u(0, x)\ \partial u/\partial t(0, x)$..., find $u(t, x)$ on the domain as governed by the equation (1), where t is to be considered as time.

The finite difference approximation to $u(t, x)$ is obtained by partitioning or discretizing the domain R and solving for the dependent variable at the mesh points by means of consistent approximations to the derivatives of (1).

III. Space Derivative Approximations

It is subsequently shown that when difference equations are used to obtain the discrete solution by means of a row-by-row calculation in time, the approximations to the space derivatives must have a "definiteness" property. That is, consider an approximation to a gth derivative [*2*]:

$$\frac{\partial^g u}{\partial x^g} = \sum_r \frac{\gamma_r u(t, x + r\Delta x)}{\Delta x^g} + 0\left(\frac{\partial^{g+1} u}{\partial t^{g+1}}\right) \tag{2}$$

where at least $g + 1$ values of r and γ_r are required. When $r = 0, 1, 2,$... this is a forward difference scheme. The central and backwards schemes of all orders of accuracy are a form of (2). The Fourier transform of (2) with respect to the space axis with k as the transform or dual variable:

$$F\left\{\frac{\partial^g u}{\partial x^g}\right\} \cong F\left\{\sum_r \frac{\gamma_r u(t, x + r\Delta x)}{\Delta x^g}\right\} = \sum_r \frac{\gamma_r e^{jkr\Delta x} U(t, k)}{\Delta x^g} \tag{3}$$

has complex coefficients which are periodic in k. Let Re{ } and Im{ } denote the real and imaginary parts of the coefficients, then:

Definition 1. An even order derivative is said to be approximated by a definite difference scheme if the imaginary part is uniquely zero and the real part of the coefficients of its Fourier transform are bounded from above or below by zero for all k. That is:

$$\mathrm{Re}\{\ \} \leqslant 0 \qquad \text{or} \quad \mathrm{Re}\{\ \} \geqslant 0.$$

The algebraic sign is positive for $g = 4, 8, 12$, and negative for $g = 2, 6, 10$.

DEFINITION 2. An odd order derivative is said to be approximated by a definite difference scheme if

$$\mathrm{Re}\{\ \} \leqslant 0 \qquad \text{or} \quad \mathrm{Re}\{\ \} \geqslant 0$$

while for the same scheme, the imaginary part may be expressed by

$$\mathrm{Im}\{\ \} = \frac{\mathrm{Sin}\, k\Delta x\, f(k\Delta x)}{\Delta x^{g}}$$

where the quantity $f(k\Delta x)$ is bounded from above or below for all k:

$$f(k\Delta x) \geqslant 0 \qquad \text{or} \quad f(k\Delta x) \leqslant 0.$$

The algebraic sign of $f(k\Delta x)$ depends on the order of derivative. The following property for the odd order derivatives is proven by construction, and has been established through the 7th order.

Property I An odd gth order derivative may be approximated by a $g + 1$ point difference scheme which may have, as desired, a negative or positive definite real Fourier transform coefficient while the quantity $f(k\Delta x)$ is definite.

The forward and backwards difference schemes for a first derivative

$$\frac{\partial u}{\partial x} \cong \frac{u(t, x + \Delta x) - u(t, x)}{\Delta x} \tag{4a}$$

$$\frac{\partial u}{\partial x} \cong \frac{u(t, x) - u(t, x - \Delta x)}{\Delta x} \tag{4b}$$

have negative and positive definite real parts respectively for their Fourier transforms.

IV. TIME DERIVATIVE APPROXIMATIONS

Let the α_i' differential operators of (1) be approximated by difference expressions and the resulting differential-difference equation be Fourier transformed with respect to the space axis to yield:

$$\frac{\partial^n U}{\partial t^n} - \alpha_{n-1} \frac{\partial^{n-1} U}{\partial t^{n-1}} \cdots - \alpha_0 u - \beta_{n-1} \frac{\partial^{n-1} U}{\partial t^{n-1}} \cdots - \beta_0 U = 0. \tag{5}$$

An example of this process is:

$$F\left\{\alpha_1' \frac{\partial u}{\partial t}\right\} = F\left\{\frac{\partial^2}{\partial x^2} \frac{\partial u}{\partial t}\right\}$$

$$= \frac{F\left\{\frac{\partial u}{\partial t}(t, x + \Delta x) - 2\frac{\partial u}{\partial t}(t, x) + \frac{\partial u}{\partial t}(t, x - \Delta x)\right\}}{\Delta x^2}$$

$$= \left(\frac{e^{jk\Delta x} - 2 + e^{-jk\Delta x}}{\Delta x}\right)\frac{\partial U}{\partial t}(t, k) = \alpha_1 \frac{\partial U}{\partial t}. \tag{6}$$

Next, the time derivatives of (5) may be approximated by difference expressions. There are a large variety of ways in which this may be accomplished. Since a change in variables will permit a difference scheme for a gth derivative to be reduced to a $g + 1$ step process, there is no loss in generality by expression $\partial^n U/\partial t^n$ of (5) by an $n + 1$ step approximation and expressing $\partial^i U/\partial t^i$ $i = 0, 1, 2,..., n - 1$ by means of a $p + 1$ step process where $p \geqslant n$. Hence (5) becomes:

$$\frac{1}{\Delta t^n}\left\{\frac{U^p}{0!} - \frac{nU^{p-1}}{1!} + \frac{n(m-1)\,U^{p-2}}{2!} \cdots (-1)^n\, U^{p-n}\right\}$$

$$- \frac{\alpha_{n-1}}{\Delta t^{n-1}}\{b_0 U^p + b_1 U^{p-1} + b_2 U^{p-2} \cdots + b_n U^{p-n} + b_{n-1} U^{p-m-1} \cdots b_p U^0\}$$

$$- \frac{\alpha_{n-2}}{\Delta t^{n-2}}\{c_0 U^p + c_1 U^{p-2} + c_2 U^{p-3} \cdots c_n U^{p-n} \cdots c_p U^0\}$$

$$\vdots$$

$$- \frac{\beta_{n-1}}{\Delta t^{n-1}}\{\qquad\} \cdots - \beta_0\{\quad\} = 0, \tag{7}$$

where the notation $U^{p-\nu} \equiv U(t + (p - \nu)\,\Delta t, k)$ has been used to clarify the expression. Note that the nth derivative has been grouped at the highest time steps. The coefficients b_ν must be such that:

$$\sum_{\nu=0}^{p} b_\nu = 0, \qquad \sum_{\nu=0}^{p} \frac{\nu b_\nu}{(n-1)} = 0,..., \sum_{\nu=0}^{p} \frac{\nu_{n-1} b_\nu}{(n-1)!} = 1 \tag{8}$$

in order to be consistent to $\partial^{n-1}U/\partial t^{n-1}$ and $p - n + 1$ degrees of freedom are available to choose the b_ν coefficients. The other approximations to $\partial^i U/\partial t^i$ must satisfy restrictions similar to (8) and $p - i$ degrees of freedom are available to adjust the coefficients. The coefficients may be chosen to insure a stable difference scheme.

V. Stability of the Difference Approximations

If dummy variables are substituted for each of the p steps in (7), this equation may be brought into vector form:

$$\mathbf{V}^{m+1} = \mathbf{G}'(\Delta t, \Delta x, k)\, \mathbf{V}^{m} \tag{9a}$$

where $\mathbf{G}'$ is a $p \times p$ matrix and an intermediate inverse matrix may have to exist. Then if $\Delta x = f(\Delta t)$ where f is continuous and such that $f(0) = 0$, then (9a) becomes

$$\mathbf{V}^{m+1} = \mathbf{G}(\Delta t, k)\, \mathbf{V}^{m}. \tag{9b}$$

It is seen that m applications of (9b) carries the initial conditions into the difference solution at $(m + 1)\,\Delta t$, hence the definition [1]:

Definition 3. A difference scheme for an initial-value problem is termed stable if for some positive τ, the matrices:

$$\mathbf{G}(\Delta t, k)^{m} \qquad \text{for} \quad \begin{cases} 0 < \Delta t < \tau \\ 0 \leqslant m\Delta t \leqslant T \\ \text{all } k \end{cases}$$

are uniformly bounded.

The stability condition which is used here is:

Theorem–*Von Neuman* [1]. *A necessary condition for stability of the difference scheme is that the eigenvalues of the matrix* $\mathbf{G}(\Delta t, k)$ *must be*:

$$|\lambda_i| \leqslant 1 + 0(\Delta t) \qquad \text{for} \quad \begin{cases} 0 < \Delta t < \tau \\ \text{all } k \\ i = 1, 2, 3,..., p. \end{cases} \tag{10}$$

If equation (7) is multiplied by Δt^n, it is seen that the characteristic equation is:

$$\begin{aligned} P(\lambda) = {} & \lambda^{p}[1 - \alpha_{n-1}b_0\Delta t - \alpha_{n-2}c_0\Delta t^2 - \alpha_{n-3}\, d_0\Delta t^3 \cdots - \beta_{n-1}r_0\Delta t \cdots] \\ & + \lambda^{p-1}[-n - \alpha_{n-1}b_1\Delta t - \alpha_{n-2}c_1\Delta t^2 - \alpha_{n-3}\, d_1\Delta t^3 \cdots - \beta_{n-1}r_1\Delta t \cdots] \\ & \vdots \\ & + \lambda^{p-n}[(-1)^n - \alpha_{n-1}b_n\Delta t - \alpha_{n-2}c_n\Delta t^2 \cdots - \beta_{n-1}r_n\Delta t \cdots] \\ & + \lambda^{p-n-1}[-\alpha_{n-1}b_{n-1}\Delta t - \alpha_{n-2}c_{n-1}\Delta t^2 \cdots - \beta_{n-1}r_{n-1}\Delta t \cdots] \\ & \vdots \\ & + \lambda^{0}[-\alpha_{n-1}b_0\Delta t - \alpha_{n-2}c_0\Delta t^2 \cdots - \alpha_{n-1}r_0\Delta t \cdots - \beta_0 r_0\Delta t^n] \\ = {} & a_p\lambda^{p} + a_{p-1}\lambda^{p-1} + a_{p-2}\lambda^{p-2} \cdots + a_0\,. \end{aligned} \tag{11}$$

The roots of this complex polynomial must satisfy (10) in order for the difference equation to be stable. The known technique for testing the roots of $p(\lambda)$ with respect to the unit circle of a complex plane is the Schur–Cohn criterion:

Schur–Cohn Criterion [3]. If for the polynomial

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 \cdots a_{p-1}\lambda^{p-1} + a_p\lambda^p \tag{12}$$

all the determinants:

$$\Delta_s = \left|\begin{array}{cccccc|cccccc}
a_0 & & & & & & a_p & a_{p-1} & a_{p-2} & a_{p-3} & \cdots & a_{p-s+1} \\
a_1 & a_0 & & & & & & a_p & a_{p-1} & a_{p-2} & & a_{p-s+2} \\
a_2 & a_1 & a_0 & & 0 & & & & a_p & a_{p-1} & & \\
a_3 & a_2 & a_1 & a_0 & & & & & & a_p & & \\
\vdots & & & & \ddots & & & & & & \ddots & \vdots \\
 & & & & a_0 & & & & 0 & & a_p\ a_{p-1} & a_{p-2} \\
 & & & & a_1 & a_0 & & & & & a_p & a_{p-1} \\
a_{s-1} & a_{s-2} & a_{s-3} & & a_2 & a_1\ \ a_0 & & & & & & a_p \\
\hline
\bar{a}_p & & & & & & \bar{a}_0 & \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_{s-} \\
\bar{a}_{p-1} & \bar{a}_p & & & & & & \bar{a}_0 & \bar{a}_1 & \bar{a}_2 & & \\
\bar{a}_{p-2} & \bar{a}_{p-1} & \bar{a}_p & & 0 & & & & \bar{a}_0 & \bar{a}_1 & & \\
\bar{a}_{p-3} & \bar{a}_{p-2} & \bar{a}_{p-1} & \bar{a}_p & & & & & & & & \\
\vdots & & & & \ddots & & & & & & \ddots & \vdots \\
 & & & & \bar{a}_p & & & & 0 & & \bar{a}_0\ \bar{a}_1 & \bar{a}_2 \\
 & & & & \bar{a}_{p-1} & \bar{a}_p & & & & & \bar{a}_0 & \bar{a}_1 \\
\bar{a}_{p-a+1} & \bar{a}_{p-s+2} & \bar{a}_{p-s+3} & & \bar{a}_{p-2} & \bar{a}_{p-1}\ \bar{a}_p & & & & & & \bar{a}_0
\end{array}\right| \tag{13}$$

$$s = 1, 2, 3, 4, \ldots p$$

are different from zero, then $p(\lambda)$ has no zeros on the unit circle $|\lambda| = 1$ and r zeros within this circle; r being the number of variations in sign of the determinant sequence $1, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \ldots, \Delta_p$. To have all the roots inside the unit circle:

$$\Delta_s < 0 \quad \text{for} \quad s = 1, 3, 5, 7, 9, \ldots, \text{ odd determinants} \tag{14}$$

$$\Delta_s > 0 \quad \text{for} \quad s = 2, 4, 6, 8, \ldots, \text{ even determinants} \tag{15}$$

In equation (13) the bar over the function denotes complex conjugate.

Note that the quadrants obtained by partitioning the $2p \times 2p$ determinant (13) are upper or lower triangular matrices. These matrices commute, and the result is a $p \times p$ Hermitian matrix whose determinant is real. The determinants for $s < p$

are the principal minors of the $p \times p$ Hermitian matrix. For example, when $p = 3$:

$$\Delta_3 = \left|\begin{array}{c|c|c} a_0\bar{a}_0 - a_3a_3 & a_0\bar{a}_1 - a_2\bar{a}_3 & a_0\bar{a}_2 - a_1\bar{a}_3 \\ \Delta_1 & & \\ \hline \bar{a}_0a_1 - \bar{a}_2a_3 & a_0\bar{a}_0 - a_3\bar{a}_3 & a_0\bar{a}_0 - a_2\bar{a}_3 \\ & +a_1\bar{a}_1 - a_2\bar{a}_2 & \\ & \Delta_2 & \\ \hline \bar{a}_0a_2 - \bar{a}_1a_3 & \bar{a}_0a_1 - \bar{a}_2a_3 & a_0\bar{a}_0 - a_3\bar{a}_3 \end{array}\right| \quad (16)$$

To examine the roots of (11) several simplifications can be made. The first of these is that the "homogeneous" part of (11) predominately fixes stability of the difference scheme.The homogeneous part is considered to be the difference approximations to:

$$\frac{\partial^n u}{\partial t^n} - \sum_{i=0}^{m-1} \alpha_i' \frac{\partial u}{\partial t^i}.$$

The terms from the β_i are "nonhomogeneous".

If the gth eigenvalue of (11) is written as:

$$\lambda_q = f_1\left(\frac{\Delta t}{\Delta x}, \frac{\Delta t^2}{\Delta x},..., \frac{\Delta t}{\Delta x^2}, \frac{\Delta t^2}{\Delta x^2} \cdots\right) + f_2(\Delta t, \Delta t^2, \Delta t^3 \cdots), \quad (17)$$

it is seen that the homogeneous part of equation (11) dominates $f_1(\ )$ and hence unity of equation (10). The nonhomogeneous terms from β_i introduce cross products in Δt or else terms in $f_2(\ )$, both of which are small and can be included in $O(\Delta t)$. This consideration of the homogeneous part extends to a system of first time order partial differential equations which are expressed in matrix form.

The second simplification in examining the stability of (11) is that the expansion of the Schur–Cohn determinants (13) yields real terms which can be arranged in ascending powers of Δt, Δt^2, Δt^3,... . The result is given as:

PROPOSITION I. *If the following conditions are met for the difference approximation to equation* (1)

(a) *definite difference schemes are used to approximate the space derivatives*

(b) *the lowest power Δt-term of each Schur–Cohn determinant of the characteristic polynomial is of the proper algebraic sign and is nonzero except at*

$$k = 0, 2\pi/\Delta x, 4\pi/\Delta x,...$$

then the difference approximation satisfies the Von Neumann necessary condition for stability as $\Delta t \to 0$ for a fixed Δx.

To see the basis for the proposition, note the use of definite difference schemes forces the lowest power Δt term in the Schur–Cohn determinants to approach zero from one direction only as k varies, $-\infty < k < \infty$. The odd derivatives contribute $\sin^{2s} k\Delta x$ as a factor, $s = 0, 1, 2, \ldots$, because each determinant is real. With these two consequences of definite difference schemes, Δt can always be made small enough such that the lowest power of Δt dominates the determinant.

Proposition I is also an "open loop" (as opposed to iterative) method to test the roots of a difference scheme. Although such a test is unnecessary for linear, constant coefficient equations, it is useful if the stability of the difference scheme must often be tested during the course of the solution.

VI. Properties of Time Polynomial Approximations

In this section the "invariants" of polynomial difference approximations are presented. The "invariants" are relationships which are true for all consistent approximations to an equation since they are limiting quantities which hold as $\Delta t \to 0$.

Consider first the three-step time difference approximation to a second order equation:

$$\frac{\partial^2 U}{\partial t^2} - \alpha_1 \frac{\partial U}{\partial t} - \alpha_0 U \cong \left[\frac{1}{\Delta t^2} + \frac{b_1\alpha_1}{\Delta t} + c_1\alpha_0\right] U^{m+1} + \left[\frac{-2}{\Delta t^2} + \frac{b_2\alpha_1}{\Delta t} + c_2\alpha_0\right] U^{m} + \left[\frac{1}{\Delta t^2} + \frac{b_3\alpha_1}{\Delta t} + c_3\alpha_0\right] U^{m-1}, \tag{18}$$

where the space dependency has been approximated and Fourier transformed prior to this step.

The characteristic polynomial of (18) is:

$$\begin{aligned} p(\lambda) &= [1 + b_1\Delta t\alpha_1 + c_1\Delta t^2\alpha_0]\,\lambda^2 \\ &\quad + [-2 + b_2\Delta t\alpha_1 + c_2\Delta t^2\alpha_0]\,\lambda + [1 + b_3\Delta t\alpha_1 + c_3\Delta t^2\alpha_0] \\ &= a_2\lambda^2 + a_1\lambda + a_0 = 0. \end{aligned} \tag{19}$$

The constraints are:

$$b_1 + b_2 + b_3 = 0, \quad 2b_1 + b_2 = -1, \qquad c_1 + c_2 + c_3 = -1. \tag{20}$$

The Schur–Cohn determinants and their proper signs are:

$$\Delta_1 = a_0\bar{a}_0 - a_2\bar{a}_2 < 0 \tag{21}$$

$$\Delta_2 = \begin{vmatrix} a_0\bar{a}_0 - a_2\bar{a}_2 & a_0\bar{a}_1 - a_1\bar{a}_2 \\ \bar{a}_0a_1 - \bar{a}_1a_2 & a_0\bar{a}_0 - a_2\bar{a}_2 \end{vmatrix} > 0. \tag{22}$$

The coefficients of equation (19) are introduced into Δ_1 and Δ_2. After eliminating b_2, b_3, and c_3 and expanding these determinants, it is seen that the lowest power Δt terms are "invariant" with respect to the remaining b_i's and c_i's which have not been specified.

Property II. The "invariants" of a three-step polynomial approximation to a second-order PDE are:

$$\Delta_1 \cong \Delta t(\alpha_1 + \bar{\alpha}_1), \tag{23}$$

$$\Delta_2 \cong \Delta t^4(\alpha_0 - \bar{\alpha}_0)^2 - \Delta t^4(\alpha_1 + \bar{\alpha}_1)(\alpha_0\bar{\alpha}_1 + \bar{\alpha}_0\alpha_1). \tag{24}$$

The invariants are the lowest terms in Δt. For example, the full expression for Δ_1 is:

$$\begin{aligned} \Delta_1 = {} & \Delta t(\alpha_1 + \bar{\alpha}_1) + \Delta t^2(-1 - c_1 - c_2)(\alpha_0 + \bar{\alpha}_0) \\ & + \Delta t^2(1 - 2b_1)\,\alpha_1\bar{\alpha}_1 \\ & + \Delta t^3(-1 - b_1 - c_1 - c_2 - 2b_1c_1 - b_1c_2)(\alpha_0\bar{\alpha}_1 + \bar{\alpha}_0\alpha_1) \\ & + \Delta t^4(1 + 2c_1 + 2c_2 + c_1^2 + 2c_1c_2)\,\alpha_0\bar{\alpha}_0\,. \end{aligned} \tag{25}$$

Property II shows the lowest order Δt terms which do not vanish for the three-step method are independent of the polynomial approximations used for $\alpha_1'\partial u/\partial t$ and $\alpha_0'u$. Thus, in Δ_1, it is seen that regardless of the derivative operators that α_1' describes, these operators must result in a negative definite real part for α_1; that is:

$$\mathrm{Re}\{\alpha_1\} \leqslant 0 \tag{26}$$

in order to have $\Delta_1 \leqslant 0$. When a definite difference scheme is used with $\mathrm{Re}\{\alpha_1\} \leqslant 0$, Δ_1 possesses the proper algebraic sign for all k, and one root of the equation remains inside the unit circle due to the fact the first Schur–Cohn criterion is satisfied. The second determinant involves the product of several terms. In (24) let:

$$\alpha_0 = a + jb, \qquad \alpha_1 = c + jd. \tag{27}$$

Then the invariant part of Δ_2 is:

$$\begin{aligned} \Delta_2 & \cong -4b^2\Delta t^4 - 2c(2ac + 2bd)\,\Delta t^4 \\ & \cong 4\Delta t^4\{-b^2 - ac^2 - bcd\}. \end{aligned} \tag{28}$$

Since Δ_1 requires c to be negative, in order that $\Delta_2 \geqslant 0$, the second condition must have $a \leqslant 0$ and/or Sign b = Sign d. The terms b and d must have Sin $k\Delta x$ as a factor if they represent a definite difference approximation. Note for $b \neq 0$, α_0 cannot have any odd order space derivative higher than the square of the largest power α_1 space derivative or else Δ_2 cannot be made positive.

Observe from equation (13) that $\Delta_1 = a_0\bar{a}_0 - a_p\bar{a}_p$ for any Schur–Cohn system of determinants, so that the most advanced time-step must necessarily also contain a term of the approximation for the highest time derivative in order to have $\Delta_1 < 0$ as $\Delta t \to 0$.

If the highest time derivative is reduced to its minimal form and grouped at the most advanced time-steps then a general theorem is the following:

PROPOSITION II. *Every $p + 1$ time-step approximation to an* n*th order equation in time has identically the same invariants for the $\Delta_{p-n+1}, \Delta_{p-n+2}, \Delta_{p-n+3}, \ldots, \Delta_p$ Schur–Cohn determinants. Only the higher order Δt terms and the $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \ldots, \Delta_{p-n}$ determinants vary with the type of approximation where $p \geqslant n$.*

The argument to support this theorem is that if the time polynomial approximations are consistent to the equation, then, in the limit as $\Delta t \to 0$ only the lowest order terms—the invariants—dominate in the final n determinants.

Consider the partial differential equation:

$$\frac{\partial^3 u}{\partial t^3} - \alpha_2' \frac{\partial^2 u}{\partial t^2} - \alpha_1' \frac{\partial u}{\partial t} - \alpha_0' u = 0. \tag{29}$$

A general four time-step approximation to this equation has the characteristic polynomial:

$$\begin{aligned} p(\lambda) = {} & \lambda^3(1 + b_1\Delta t\alpha_2 + c_1\Delta t^2\alpha_1 + d_1\Delta t^3\alpha_0) \\ & + \lambda^2(-3 + b_2\Delta t\alpha_2 + c_2\Delta t^2\alpha_1 + d_2\Delta t^3\alpha_0) \\ & + \lambda(3 + b_3\Delta t\alpha_2 + c_3\Delta t^2\alpha_1 + d_3\Delta t^3\alpha_0) \\ & + (-1 + b_4\Delta t\alpha_2 + c_4\Delta t^2\alpha_1 + d_4\Delta t^3\alpha_0). \end{aligned} \tag{30}$$

The constraints which must be satisfied by the coefficients in the polynomial are:

$$\begin{array}{lll} d_1 + d_2 + d_3 + d_4 = -1 & c_1 + c_2 + c_3 + c_4 = 0 & b_1 + b_2 + b_3 + b_4 = 0 \\ & 3c_1 + 2c_2 + c_3 = -1 & 3b_1 + 2b_2 + b_3 = 0 \\ & & 9b_1 + 4b_2 + b_3 = -1. \end{array} \tag{31}$$

The constraints are used to eliminate d_4, c_3, c_4, b_2, b_3, and b_4, and the Schur–Cohn determinants are given by the principal minors of equation (16). When the determinants of equation (16) are expanded, *the invariants of a third-order equation are*:

$$\Delta_1 \cong \Delta t(\alpha_2 + \bar{\alpha}_2). \tag{32}$$

$$\Delta_2 \cong \Delta t^4(\alpha_1 - \bar{\alpha}_1)^2 - \Delta t^4(\alpha_2 + \bar{\alpha}_2)(\alpha_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_2) - \Delta t^4(\alpha_2 + \bar{\alpha}_2)(\alpha_0 + \bar{\alpha}_0). \tag{33}$$

$$\begin{aligned}\Delta_3 \cong\ & \Delta t^9(\alpha_0 + \bar{\alpha}_0)^3 + \Delta t^9(\alpha_2 + \bar{\alpha}_2)(\alpha_0\bar{\alpha}_1 + \bar{\alpha}_0\alpha_1)(\alpha_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_2) \\ & + 1/4\Delta t^9(\alpha_0 + \bar{\alpha}_0)^2(\alpha_2 + \bar{\alpha}_2)(\alpha_2 - \bar{\alpha}_2)^2 - \Delta t^9/2(\alpha_0 + \bar{\alpha}_0)(\alpha_0 - \bar{\alpha}_0) \\ & \times (\alpha_2 + \bar{\alpha}_2)^2(\alpha_2 - \bar{\alpha}_2) - \Delta t^9(\alpha_1 - \alpha_1)^2(\alpha_0\bar{\alpha}_1 + \bar{\alpha}_0\alpha_1) \\ & + 2\Delta t^9(\alpha_0 + \bar{\alpha}_0(\alpha_2 + \bar{\alpha}_2)(\alpha_0\bar{\alpha}_1 + \bar{\alpha}_0\alpha_1) + \Delta t^9/2(\alpha_0 - \bar{\alpha}_0)(\alpha_1 - \bar{\alpha}_1)^3 \\ & - \Delta t^9/2(\alpha_0 - \bar{\alpha}_0)(\alpha_0 + \bar{\alpha}_0)(\alpha_1 - \bar{\alpha}_1)(\alpha_2 + \bar{\alpha}_2) + \Delta t^9/2(\alpha_0 - \bar{\alpha}_0)^2(\alpha_2 + \bar{\alpha}_2)^2.\end{aligned} \tag{34}$$

Note the similarity between the first and second invariants above with the second-order invariants. In equation (33) one new term $-\Delta t^4(\alpha_2 + \bar{\alpha}_2)(\alpha_0 + \bar{\alpha}_0)$ is present which does not appear for the second-order equation. (The similarity in Δ_1 and Δ_2 of the second- and third-order equations suggests that there might be a pattern for the *n*th order equation from which all lower order equation invariants are derived.)

Note in equation (25) for the second-order case, that when the invariant term is not present, the roots of the amplification matrix for the difference scheme depend on the variable cœfficients in the polynomial approximation. In the absence of invariants, it is possible to choose the unspecified coefficients of the polynomial to obtain a stable formulation. However, when the terms in the partial differential equation yield incorrect signs for the invariants a time polynomial cannot be used except in one circumstance. This circumstance is: When every Schur–Cohn determinant is positive all roots lie outside the unit circle. In this case the difference equation is solved for the lowest time step and the values of the mesh are solved for row-by-row backward in time. This is a final value type problem of which the backwards heat equations is an example:

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}. \tag{35}$$

VII. Illustrative Examples

In this section several examples of mixed initial and boundary value problems are worked out in detail. The Schur–Cohn determinants are utilized to choose the spatial derivatives approximations and the time polynomials, and demonstrate the preceding methods.

EXAMPLE I—A QUASI-LINEAR VIBRATING BEAM

Consider the vibrating beam equation with quasi-linear coefficients:

$$\frac{\partial^2 u}{\partial t^2} = -(u)^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} = \alpha_0' u. \tag{Ia}$$

Because of the nonconstant nature of the coefficients Δt may have to be modified during the computation. The first and second derivatives for variable Δt are

$$\frac{\partial u}{\partial t} \cong \frac{u_n^{m+1} - u_n{}^m}{\Delta t_1} \qquad \frac{\partial^2 u}{\partial t^2} \cong \left(\frac{u_n^{m+2} - u_n^{m+1}}{\Delta t_1 \Delta t_2}\right) - \left(\frac{u_n^{m+1} - u_n{}^m}{\Delta t_1 \Delta t_1}\right). \tag{Ib}$$

Using these formulas, a three time-step approximation to equation (Ia) is:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \alpha_0' u = 0 \rightarrow [1 + \Delta t_2 \Delta t_1 c_1 \alpha_0]\, H^{m+2} \\ + [-1 - \Delta t_2/\Delta t_1 + \Delta t_2 \Delta t_1 c_2 \alpha_0]\, H^{m+1} \\ + [\Delta t_2/\Delta t_1 + \Delta t_2 \Delta t_1 c_3 \alpha_0]\, H^m \end{aligned} \tag{Ic}$$

where $c_1 + c_2 + c_3 = -1$. When the coefficient c_3 is eliminated by the constraint, the first Schur–Cohn determinant for the characteristic equation is:

$$\begin{aligned} \Delta_1 = \left(\frac{\Delta t_2}{\Delta t_1}\right)^2 - 1 - (\Delta t_2)^2 (1 + 2c_1 + 2_2)(\alpha_0 + \bar{\alpha}_0) \\ + (\Delta t_2 \Delta t_1)^2 (1 + 2c_1 + 2c_2 + 2c_1 c_2 + c_2{}^2)\, \alpha_0 \bar{\alpha}_0 . \end{aligned} \tag{Id}$$

Note the absence of the first invariant which is consequence of $\alpha_1' = 0$.

Since α_0' contains only even order derivatives, the choice of space derivative difference schemes is fixed, and it can be seen that these schemes have only negative definite real parts. If Δ_1 is to have the proper algebraic sign, then:

$$2c_1 + c_2 < -1. \tag{Ie}$$

In order to avoid an implicit formulation for H^{m+2}, use the following quantities:

$$\begin{aligned} c_2 &= -2 \\ c_1 &= 0 \end{aligned} \tag{If}$$

which forces the characteristic equation of (Ic) to be:

$$p(\lambda) = \lambda^2 + \lambda \left(-1 - \frac{\Delta t_2}{\Delta t_1} - 2\Delta t_1 \Delta t_2 \alpha_0\right) + \left(\frac{\Delta t_2}{\Delta t_1} + \Delta t_2 \Delta t_1 \alpha_0\right). \tag{Ig}$$

Using the values from equation (If), equation (Id) is:

$$\Delta_1 = \left(\frac{\Delta t_1}{\Delta t_2}\right)^2 - 1 + (\Delta t_2)^2 (\alpha_0 + \bar{a}_0) + (\Delta t_2 \Delta t_1)^2 \alpha_0 \bar{a}_0 . \tag{Ih}$$

The second Schur–Cohn determinant for a *fixed* time-step size using the above values of c_1 and c_2 is:

$$\Delta_2 = \Delta t^4(\alpha_0 - \bar{\alpha}_0)^2 - 2\Delta^6\alpha_0\bar{\alpha}_0(\alpha_0 + \bar{\alpha}_0) + 9\Delta t^8\alpha_0{}^2\bar{\alpha}_0{}^2. \tag{Ii}$$

It is seen in equation (Ih) that unless $\Delta t_2 \leqslant \Delta t_1$, $\Delta_1 > 0$ for values of $k\Delta x \cong 0, 2\pi, 4\pi,...$ instabilities are introduced into the calculation. However, if these disturbances due to Δt alone are sufficiently small the Von Neumann stability criterion may still be satisfied. Therefore, if the time-step changes are mild, the Schur–Cohn determinants are calculated with a *fixed* time-step.

The boundary conditions used for (Ia) are:

$$u(t, 0) = \frac{\partial u}{\partial x}(t, 0) = u(t, 1) = \frac{\partial u}{\partial x}(t, 1) = 0 \tag{Ij}$$

and the initial values:

$$u(0, x) = \frac{256}{9} x^2(1 - x)^2 \operatorname{Sin} 2\pi x, \qquad \frac{\partial u}{\partial t}(0, x) = 0. \tag{Ik}$$

The initial conditions must be compatible with the boundary conditions at $x = 0$ and $x = 1$, hence the unusual form of the state, $u(0, x)$. Notice that the initial condition has odd symmetry about $x = .5$.

The candidate time-step size, Δt_2 must satisfy

$$1/2(\Delta t_2{}^2)\, |(\alpha_0 + \bar{\alpha}_0)|_{\substack{\max k_m \\ \max u_n}} = (\Delta t_2)^2 \left| \operatorname{Re} \left\{ -(u_n{}^m)^2 \frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2} \right\} \right|_{\substack{\max k_m \\ \max u_n}} < 1. \tag{Il}$$

Notice that all terms of Δ_2 are positive, so only Δ_1 must be satisfied by a sufficiently small Δt step. The fourth derivative and the second derivative take on a maximum value for the space frequency $k = \pi/\Delta x$. Hence, if equation (Il) is satisfied at this value of space frequency, Proposition I does not have to be applied for all values of k in order to test the Schur-Cohn determinants.

Equation (Il) was satisfied during the computation by storing the maximum value of u found in the mth row in order to calculate the step size for the $(m + 1)$th row. The results of the numerical solution are presented in Figure 1 and Figure 2.

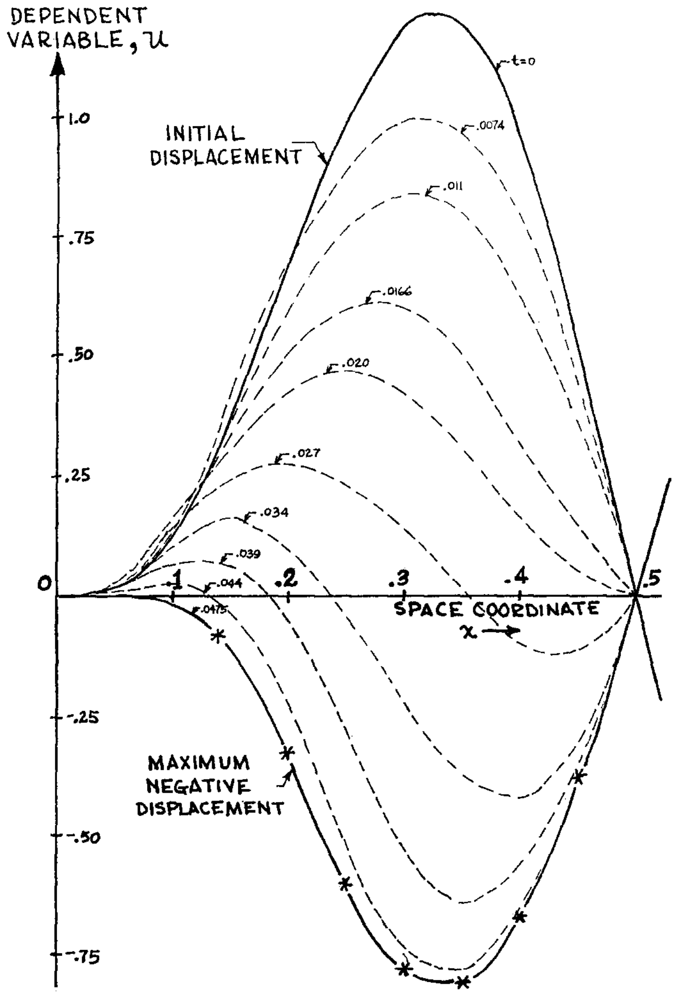

FIG. 1. Solution to example I for varying time.

In Figure 2 the time-step sizes and the interval over which they were used are indicated. It can be seen in this figure that as the magnitude of the dependent variable changes, the step size does accordingly.

A total of 25 steps were taken for the solution to reach $t = .01$ with the variable method. It is of interest to know the accuracy of such a method and for this purpose the data of Table I were prepared. In this table several step sizes $\Delta t = .0001$, $\Delta t_m = \Delta t_{m-1} + .000001$, and $\Delta t = .0002$ are compared against the variable step method.

TABLE I

COMPARISON OF SOLUTIONS TO EXAMPLE I FOR VARIOUS TIME-STEP SIZES TIME t = .010

Space Coordinate	Δt = .0001 100 Steps	$\Delta t_m = \Delta t_{m-1}$ + .000001[a] 73 Steps	Δt = .0002 50 Steps	Variable Δt 25 Steps
0	0	0	0	0
.05	0	0	0	0
.10	.142479	.142304	.142247	.141448
.15	.413499	.413020	.413305	.410235
.20	.648381	.649757	.650741	.655771
.25	.807888	.811518	.814558	.828327
.30	.876309	.881295	.885435	.904830
.35	.836471	.841922	.846196	.867121
.40	.673310	.678363	.682067	.700624
.45	.384862	.388236	.390656	.402554
.50	−.000013	−.000013	−.000013	−.000013

[a] (Δt_0 = .0001).

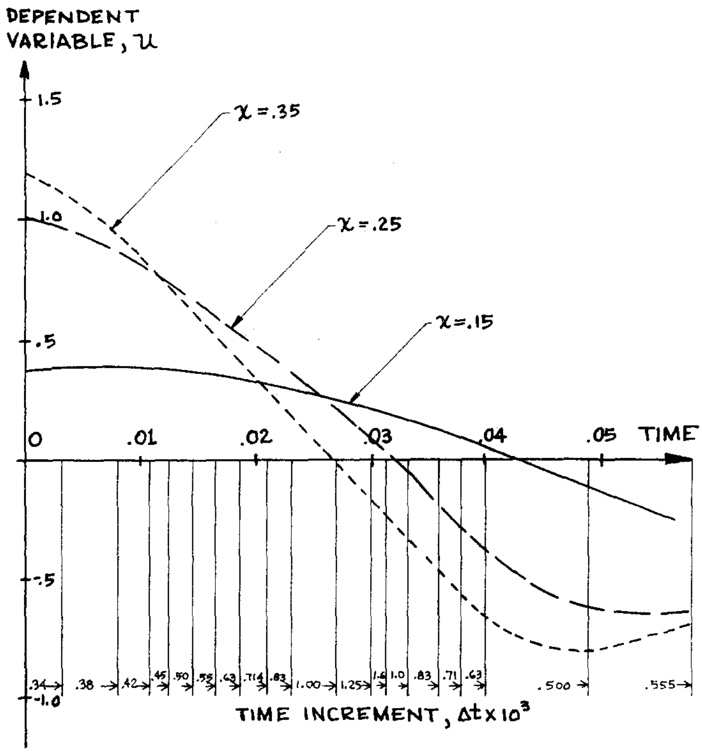

FIG. 2. Solution to example I using variable time step. Δx = .05.

The results in this table change monotonically with the step size, indicating that the variations are predominately due to truncation errors. The second column for $\Delta t_m = \Delta t_{m-1} + .000001$ is an attempt to force the next time-step to be larger than its predecessor, but it does not appear to result in an adverse effect on the solution for such small changes.

The variable step method, using one-fourth as many steps as for $\Delta t = .001$, remains within four per cent of the values calculated for this smaller step over the entire space interval. Clearly, this is adequate for most engineering applications.

A linear problem very similar to this example is presented in [1], page 185. The important distinction is that equations (Id), (Ie), and (If) have fixed the difference method on a systematic basis which does not require intuition to find a stable scheme.

Example II—A Third Order System

To demonstrate the methods of analysis which have been derived for a higher order system, consider the third-order quasi-linear equation:

$$\frac{\partial^3 u}{\partial t^3} = (1 - 2t)\, u^2 \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^3 u}{\partial x^2 \partial t} + (1 - 4x^2) \frac{\partial^5 u}{\partial x^3 \partial t^2} + \left(\frac{\partial u}{\partial x}\right) \frac{\partial^3 u}{\partial x \partial t^2}. \tag{IIa}$$

For the domain:

$$R : \{0 \leqslant t \leqslant 3, \quad 0 \leqslant x \leqslant 1\} \tag{IIb}$$

it is seen that two coefficients in equation (IIa) change sign due to the independent variables. Since these are odd order space derivatives, it is possible to choose a spatial difference expression to insure that their real parts are negative definite.

The boundary and initial conditions are

$$\begin{aligned} u(0, x) &= 0, & u(t, 0) &= 0, \\ \frac{\partial u}{\partial t}(0, x) &= x^2(1 - x)^2, & \frac{\partial u}{\partial x}(t, 0) &= 0, \\ \frac{\partial^2}{\partial t^2}(0, x) &= 0, & u(t, 1) &= 0. \end{aligned} \tag{IIc}$$

Equation (IIa) can be put into a more familiar form:

$$\frac{\partial^3 u}{\partial t^3} - \alpha_2' \frac{\partial^2 u}{\partial t^2} - \alpha_1' \frac{\partial u}{\partial t} - \alpha_0' u = 0 \tag{IId}$$

where the α_i' have the definitions:

$$\alpha_0' \equiv (1 - 2t)\, u^2 \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2},$$
$$\alpha_1' \equiv \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2}{\partial x^2}, \tag{IIe}$$
$$\alpha_2' \equiv (1 - 4x^2) \frac{\partial^3}{\partial x^3} + \left(\frac{\partial u}{\partial x}\right) \frac{\partial}{\partial x}.$$

It is evident that the α_i' are present throughout most of the solution, so that stability of the difference formulation will depend largely upon the invariants for a third-order equation given by equations (32), (33), and (34).

The first invariant is:

$$\Delta_1 \cong \Delta t(\alpha_2 + \bar{\alpha}_2) \tag{IIf}$$

and must be negative for one root to lie inside the unit circle. This is satisfied if $\mathrm{Re}\{\alpha_2\} \leqslant 0$, which implies that two separate difference schemes must be used to describe α_2 on the interval $0 \leqslant x \leqslant 1$.

If the point being calculated has space coordinate $n\Delta x < 1/2$ the difference scheme for the third derivative with respect to the space coordinate is:

$$(1 - 4x^2) \frac{\partial^3 u}{\partial x^3} \cong (1 - 4x^2) \left(\frac{1}{\Delta x^3}\right) \{-1/5 u_{n-3}^m + u_{n-1}^m - u_n^m + 1/5 u_{n+2}^m\} \tag{IIg}$$

and if $n\Delta x > 1/2$ the scheme used is:

$$(1 - 4x^2) \frac{\partial^3 u}{\partial x^3} \cong (1 - 4x^2) \left(\frac{1}{\Delta x^3}\right) \{-1/10 u_{n-3}^m + u_n^m - 3/2 u_{n+1}^m + 3/5 u_{n+2}^m\}. \tag{IIh}$$

These difference schemes satisfy the condition $\mathrm{Re}\{\alpha_2\} \leqslant 0$ and are examples of Property I.

It is clear that these schemes require the used of artificial points off the boundary. These points are here treated as a smooth extension across the boundary.

$$\left(\frac{\partial u}{\partial x}\right) \frac{\partial}{\partial x} \tag{IIi}$$

uses a forward difference scheme if $\partial u/\partial x > 0$ and a backwards scheme if $\partial u/\partial x < 0$.

When the values of α_i are substituted into the expressions for the invariants-equations (32), (33), and (34)-the difference expressions will result in a stable calculation on "most of the mesh." It is apparent that at $x = 1/2$ the third derivative term of α_2 is zero, so that serious difficulties could occur near this line.

To solve for the numerical values, the changes in the difference methods are built into the computer program as switches. The $(m+1)$th time increment for a variable time step technique is calculated for the criterion:

$$\begin{matrix} m+1 \\ \text{Time:} \\ \text{Step} \end{matrix} \left\{ \begin{aligned} &(\Delta t_{m+1}) \mid \operatorname{Re} \alpha_{2n}^{m} \Big|_{\substack{\max k \\ \max x}} < 1 \\ &(\Delta t_{m+1})^2 \mid \operatorname{Re} \alpha_{1n}^{m} \Big|_{\substack{\max k \\ \max x}} < 1 \\ &(\Delta t_{m+1})^3 \mid \operatorname{Re} \alpha_{0n}^{m} \Big|_{\substack{\max k \\ \max x}} < 1. \end{aligned} \right. \tag{IIj}$$

That is, the largest values for α_i are stored from the mth time step, then the maximum for all values of k is determined. The value of Δt_{m+1} which satisfies the criterion is used as the next time-step. This method tends to force the invariants to dominate and is a negligible computation cost. By this method it is not assured that the Schur–Cohn determinants are satisfied for all values of space coordinate and all values of space frequency k.

Figure 3 displays the solution to equation (63) where it is evident that the solution grows very rapidly. For time greater than $t = .75 \times 10^{-3}$, the time-step

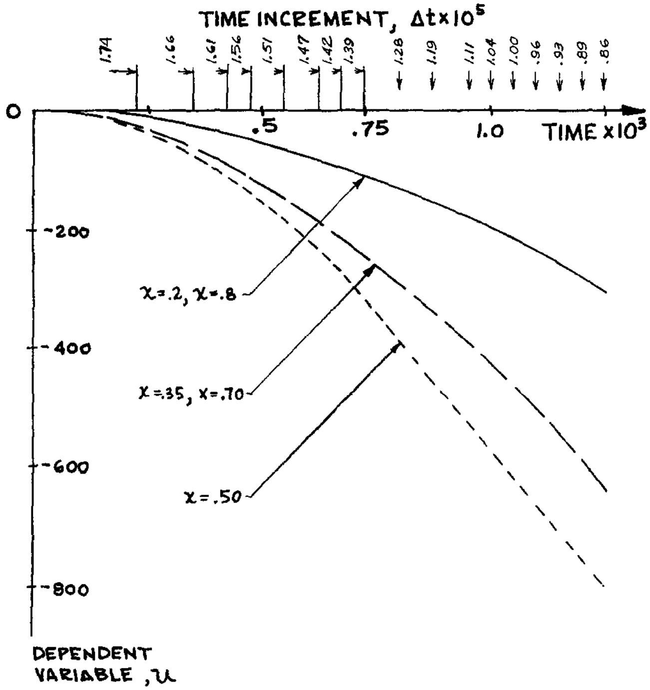

FIG. 3. Solution to example II using variable time step.

TABLE II

NUMERICAL RESULTS FOR CONSTANT STEP AND VARIABLE STEP METHODS OF EXAMPLE II, $t = 1 \times 10^{-3}$

Space Co-Cordinate	Variable time step method 63 Steps	Fixed time step $\Delta t = 1 \times 10^{-5}$ 100 Steps
.1	$-.568216 \times 10^2$	$-.560019 \times 10^2$
.15	$-.125347 \times 10^3$	$-.123614 \times 10^3$
.20	$-.201707 \times 10^3$	$-.199029 \times 10^3$
.25	$-.280954 \times 10^3$	$-.277390 \times 10^3$
.30	$-.360464 \times 10^3$	$-.356113 \times 10^3$
.35	$-.437683 \times 10^3$	$-.432677 \times 10^3$
.40	$-.509752 \times 10^3$	$-.504280 \times 10^3$
.45	$-.572863 \times 10^3$	$-.567281 \times 10^3$
.50	$-.572997 \times 10^3$	$-.566659 \times 10^3$
.55	$-.534742 \times 10^3$	$-.528561 \times 10^3$
.60	$-.479677 \times 10^3$	$-.473988 \times 10^3$
.65	$-.414241 \times 10^3$	$-.409192 \times 10^3$
.70	$-.342170 \times 10^3$	$-.337863 \times 10^3$
.75	$-.266331 \times 10^3$	$-.262883 \times 10^3$
.80	$-.189952 \times 10^3$	$-.187439 \times 10^3$
.85	$-.117349 \times 10^3$	$-.115765 \times 10^3$
.90	$-.549360 \times 10^2$	$-.128874 \times 10^2$
.95	$-.130616 \times 10^2$	$-.128874 \times 10^2$

is almost continuously reduced by the criterion (IIj) for each successive row. The space increment used is $\Delta x = .05$, so the time-step is reduced from Δt_0, where:

$$\Delta t_0/\Delta x^3 < 1 \qquad \Delta t_0 < (.05)^3 = 1.25 \times 10^{-4} \tag{IIk}$$

until the conditions of equation (IIj) are satisfied. In Table II a comparison is made at time $t = 1 \times 10^{-3}$ between a conservative constant time-step size and the results of the variable step method. This table shows very favorable agreement for the two methods at this time. Notice for $t > 1 \times 10^{-3}$ the fixed step method would be unstable since a step size smaller than the fixed increment is needed.

During the course of the solution to this problem, the Schur–Cohn determinants are observed at every tenth row in time. The value of the determinants are calculated for various space frequencies k and locations on the x coordinate. Table III presents Δ_1, Δ_2, and Δ_3 for $x = .1, .5, .6$, and .9, and is typical of the results for other parts of the mesh. From Table III it is seen that the signs $-$, $+$, $-$ are not everywhere satisfied by the determinant sequence Δ_1, Δ_2, Δ_3. This means

TABLE III

$\Delta_1, \Delta_2, \Delta_3$ VERSUS $K\Delta X$ FOR EXAMPLE II—VARIABLE STEP, $t = 1.04 \times 10^{-3}$

	$X = .1$			$X = .5$		
$K\Delta X$	Δ_1	Δ_2	Δ_3	Δ_1	Δ_2	Δ_3
0	$+.395 \times 10^{-8}$	0	0	$-.181 \times 10^{-11}$	$-.145 \times 10^{-10}$	$+.741 \times 10^{-13}$
.25	$+.160 \times 10^{3}$	$+.233 \times 10^{5}$	$-.138$	$-.498 \times 10^{-2}$	$+.780 \times 10^{-7}$	$-.294 \times 10^{-12}$
.50	$+.145 \times 10^{3}$	$+.190 \times 10^{5}$	$-.547 \times 10^{-1}$	$-.199 \times 10^{-1}$	$+.487 \times 10^{-5}$	$-.880 \times 10^{-12}$
.75	$+.211 \times 10^{2}$	$+.305 \times 10^{3}$	$-.430 \times 10^{-4}$	$-.446 \times 10^{-1}$	$+.532 \times 10^{-4}$	$+.617 \times 10^{-12}$
1.00	$+.958 \times 10^{1}$	$+.458 \times 10^{2}$	$-.181 \times 10^{-5}$	$-.788 \times 10^{-1}$	$+.280 \times 10^{-3}$	$+.611 \times 10^{-12}$
1.25	$+.411 \times 10^{1}$	$+.344 \times 10^{1}$	$-.394 \times 10^{-7}$	$-.121$	$+.976 \times 10^{-3}$	$-.454 \times 10^{-11}$
1.50	$+.161 \times 10^{1}$	$-.896$	$+.328 \times 10^{-8}$	$-.170$	$+.257 \times 10^{-2}$	$-.181 \times 10^{-10}$
1.75	$+.538$	$-.651$	$+.107 \times 10^{-8}$	$-.224$	$+.552 \times 10^{-2}$	$-.451 \times 10^{-10}$
2.00	$-.320 \times 10^{-1}$	$-.223$	$+.192 \times 10^{-9}$	$-.278$	$+.100 \times 10^{-1}$	$-.104 \times 10^{-9}$
2.25	$-.291$	$-.239 \times 10^{-1}$	$+.161 \times 10^{-10}$	$-.328$	$+.158 \times 10^{-1}$	$-.173 \times 10^{-9}$
2.50	$-.440$	$+.970 \times 10^{-1}$	$-.566 \times 10^{-10}$	$-.371$	$+.222 \times 10^{-1}$	$-.248 \times 10^{-9}$
2.75	$-.517$	$+.162$	$-.932 \times 10^{-10}$	$-.403$	$+.276 \times 10^{-1}$	$-.300 \times 10^{-9}$
3.00	$-.556$	$+.193$	$-.103 \times 10^{-9}$	$-.420$	$+.309 \times 10^{-1}$	$-.300 \times 10^{-9}$
3.25	$-.558$	$+.195$	$-.103 \times 10^{-9}$	$-.421$	$+.311 \times 10^{-1}$	$-.254 \times 10^{-9}$
	$X = .6$			$X = .9$		
0	$-.181 \times 10^{-11}$	$+.145 \times 10^{-10}$	$-.246 \times 10^{-10}$	$-.363 \times 10^{-11}$	0	0
.25	$+.727 \times 10^{-3}$	$-.437 \times 10^{-6}$	$-.327 \times 10^{-14}$	$-.495 \times 10^{-2}$	$+.562 \times 10^{-7}$	$-.206 \times 10^{-14}$
.50	$+.290 \times 10^{-2}$	$-.268 \times 10^{-4}$	$+.588 \times 10^{-13}$	$-.170 \times 10^{-1}$	$+.283 \times 10^{-5}$	$-.322 \times 10^{-12}$
.75	$+.551 \times 10^{-2}$	$-.278 \times 10^{-3}$	$-.464 \times 10^{-13}$	$-.298 \times 10^{-1}$	$+.213 \times 10^{-4}$	$-.323 \times 10^{-12}$
1.00	$+.437 \times 10^{-2}$	$-.131 \times 10^{-2}$	$+.144 \times 10^{-11}$	$-.400 \times 10^{-1}$	$+.705 \times 10^{-4}$	$-.344 \times 10^{-13}$
1.25	$-.889 \times 10^{-2}$	$-.372 \times 10^{-2}$	$+.863 \times 10^{-11}$	$-.559 \times 10^{-1}$	$+.218 \times 10^{-3}$	$+.944 \times 10^{-13}$
1.50	$-.448 \times 10^{-1}$	$-.667 \times 10^{-2}$	$+.289 \times 10^{-10}$	$-.100$	$+.111 \times 10^{-2}$	$+.859 \times 10^{-12}$
1.75	$-.111$	$-.600 \times 10^{-2}$	$+.401 \times 10^{-10}$	$-.198$	$+.544 \times 10^{-2}$	$+.298 \times 10^{-11}$
2.00	$-.208$	$+.592 \times 10^{-2}$	$-.621 \times 10^{-10}$	$-.356$	$+.190 \times 10^{-1}$	0
2.25	$-.323$	$+.359 \times 10^{-1}$	$-.477 \times 10^{-9}$	$-.550$	$+.481 \times 10^{-1}$	$+.272 \times 10^{-11}$
2.50	$-.438$	$+.831 \times 10^{-1}$	$-.133 \times 10^{-8}$	$-.737$	$+.919 \times 10^{-1}$	0
2.75	$-.532$	$+.134$	$-.246 \times 10^{-8}$	$-.878$	$+.138$	$-.727 \times 10^{-11}$
3.00	$-.585$	$+.168$	$-.332 \times 10^{-8}$	$-.952$	$+.168$	$-.525 \times 10^{-11}$
3.25	$-.588$	$+.171$	$-.338 \times 10^{-8}$	$-.957$	$+.170$	$-.181 \times 10^{-11}$

that at times roots of the characteristic equation fall outside the unit circle, but the damping of the overall mesh is sufficient to produce an apparently stable computation. A "majority" of the mesh points and frequencies have signs which are correct, but a quantitative measure of what this "majority" should be is not yet available.

XI. Conclusions

A systematic method of formulating difference expressions to initial value type partial differential equations has been presented. The Schur–Cohn criterion has demonstrated the need for definite difference schemes and provided a "fast" open loop technique to find the largest time step compatible with stability. The Schur–Cohn criterion has shown that multistep time polynomials must satisfy the "invariant" quantities which hold for all time polynomial formulations.

This paper has presented an efficient method to numerically solve quasi-linear partial differential equations. It is hoped this work also provides additional insight into finite difference formulations of mixed initial and boundary value problems.

References

1. R. D. Richtmyer, "Difference Methods for Initial-Value Problems." Interscience Publishers, New York, 1957, or the second edition by R. D. Richtmyer and K. W. Morton, Interscience Publishers, New York, 1967.
2. M. G. Salvadori and M. L. Baron, "Numerical Methods in Engineering." Prentice-Hall, Englewood Cliffs, New Jersey, 1952.
3. M. Marden, "The Geometry of the Zeros." American Mathematical Society, New York, 1949.